The problem of causality for a classical extended electron

1987 J. Phys. A: Math. Gen. 20 5885

(http://iopscience.iop.org/0305-4470/20/17/022)

J. Phys. A: Math. Gen. 20 (1987) 5885-5898. Printed in the UK

The problem of causality for a classical extended electron

R Blanco
Departamento de Física Teórica, Universidad de Cantabria, Santander, Spain

Received 3 December 1986, in final form 24 June 1987

Abstract. We show that the rigid spherically symmetric model of the electron is causal in the sense that (i) an external force acting at a time t does not influence the motion before t, and (ii) the motion before a certain time t_0 and the system of forces determine uniquely the motion after t_0. We also make some comments concening the runaway and pre-acceleration phenomena in connection with previous results. We show that the non-uniqueness of the general solution, as well as the apparent existence of pre-acceleration for certain values of the radius, are due to the existence of non-trivial solutions for the free case.

1. Introduction

The use of extended models for the electron in classical electrodynamics can be traced back to the very origin of the theory (see, for instance, Erber 1961, Hirosige 1965, Lorentz 1952, Miller 1976). However, the general tendency this century in physics has caused these models to be unknown to most physicists. Indeed, although several papers on this subject have been published this century, we can say that general results are rather scarce. Undoubtedly, the most general study can be found in Nodvik (1964).

Apart from this, the analyses appearing in the literature have been restricted to spherically symmetric charge distributions and consider separately either translational (Alvarez-Estrada and Ros Martínez 1981, Blanco *et al* 1986, Bohm and Weinstein 1948, Caldirola 1956, de la Peña *et al* 1982, França *et al* 1978, Grandy and Aghazadeh 1982, Kaup 1966, Levine *et al* 1977, Markov 1946, Moniz and Sharp 1977) or rotational non-relativistic motion (Daboul 1975, Daboul and Jensen 1973, Jiménez *et al* 1985, Rañada and Vázquez 1984). Furthermore, most works deal with one particular charge distribution. In this context we can say that the results existing for these models are rather weak. For instance, the problem of runaways is generally restricted to considering exponential solutions (Blanco *et al* 1986, Bohm and Weinstein 1948, de la Peña *et al* 1982, França *et al* 1978, Grandy and Aghazadeh 1982, Moniz and Sharp 1977). Perhaps the most general result for this problem is due to Kaup (1966) which gives sufficient conditions for the non-existence of general runaway behaviour. However, this analysis is restricted to forces not dependent on position and velocity. Another interesting point is the pre-acceleration phenomenon that has been considered in some papers (Blanco *et al* 1986, França *et al* 1978, Kaup 1966), a new consideration of which appears in the present paper, already described in Blanco *et al* (1986) for a particular (Yukawa-type) charge distribution. A question that has not been studied in the literature is the unicity of solutions starting from a given time. This problem, which is very easily solved for the Yukawa case (Blanco *et al* 1986), is solved in the present paper for almost any physically reasonable charge distribution. Finally, one important result for the free case with rather general conditions can be found in Alvarez-Estrada and Ros Martínez (1981) and slightly generalised in Blanco *et al* (1986).

All these points are clearly related to the problem of causality which is the theme of the present paper. Our aim is to clarify some of the abovementioned aspects for the case of non-relativistic translational motion of a rigid spherically symmetric charge.

After explaining the model and some of its preliminary characteristics in § 2, we show in § 3 our main result: the uniqueness of solutions starting from a finite time. Section 4 is devoted to presenting the actual knowledge on the solutions of the homogeneous equation . In § 5 we analyse in detail the problem of pre-acceleration in order to clarify previous results (de la Peña *et al* 1982, França *et al* 1978). We also make a few comments on the runaway behaviour.

2. The model

If $e\rho(r)$ denotes the charge density as a function of the distance to the centre of the charge we consider the following new functions:

$$\hat{\rho}(\omega)=\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3\xi\,\rho(\xi)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\xi}) \qquad \omega=c|\boldsymbol{k}| \tag{2.1}$$

$$H(t)=\frac{32\pi^2e^2}{3c^3}\int_0^\infty \mathrm{d}\omega\,\omega\hat{\rho}^2(\omega)\sin\omega t \tag{2.2a}$$

$$=\tfrac{8}{3}\pi e^2 t\int \mathrm{d}^3\xi\,\rho(\xi)\rho(|\boldsymbol{r}+\boldsymbol{\xi}|) \qquad |\boldsymbol{r}|=ct. \tag{2.2b}$$

Note that $\hat{\rho}$, the Fourier transform of ρ, is also spherically symmetric.

If the charge is subjected to an external force $\boldsymbol{F}(\boldsymbol{r},t)$, its motion is ruled by the following integrodifferential equation (de la Peña *et al* 1982, França *et al* 1978, Kaup 1966):

$$m\ddot{\boldsymbol{r}}(t)=\boldsymbol{F}_{\mathrm{ef}}-\int_{-\infty}^{t}\mathrm{d}t'\,H(t-t')[\ddot{\boldsymbol{r}}(t')-\ddot{\boldsymbol{r}}(t)] \tag{2.3}$$

where $\boldsymbol{r}$ is the position of the centre of the charge, m is the observable electron mass which turns out to be

$$m=m_0+m_{\mathrm{e}} \tag{2.4}$$

m_0 being the mechanical mass and m_{e} the electromagnetic mass, and $\boldsymbol{F}_{\mathrm{ef}}$ represents the effect of the external force upon the charge distribution. For example, if $\boldsymbol{F}$ is due to an electric field, we have

$$\boldsymbol{F}_{\mathrm{ef}}(\boldsymbol{r},t)=\int \mathrm{d}^3\xi\,\rho(\xi)\boldsymbol{F}(\boldsymbol{r}+\boldsymbol{\xi},t). \tag{2.5}$$

For most purposes $\boldsymbol{F}$ is such that its variation along the dimensions of the charge can be taken as negligible and we can make the approximation

$$\boldsymbol{F}_{\mathrm{ef}}\simeq\boldsymbol{F}. \tag{2.6}$$

However, this is not always necessary as long as both functions turn out to depend on $\boldsymbol{r}$ and t. Moreover, important cases include rapidly varying fields (Blanco *et al* 1986) and then care must be taken about the approximation (2.6).

As concerns equation (2.3), it is worth noticing that only the non-relativistic and dipolar approximations are considered (for more details see França *et al* (1984)).

Introducing the following quantities:

$$m_1 = m - \int_0^\infty H(t)\,\mathrm{d}t \tag{2.7}$$

$$\gamma(t) = (1/m_1)H(t) \tag{2.8}$$

we can write (2.3) as

$$\ddot{\boldsymbol{r}}(t) = \frac{\boldsymbol{F}_{\mathrm{ef}}}{m_1} - \int_{-\infty}^{t} \mathrm{d}t'\,\gamma(t-t')\ddot{\boldsymbol{r}}(t'). \tag{2.9}$$

It is convenient to define

$$\delta m = \int_0^\infty H(t)\,\mathrm{d}t \tag{2.10}$$

$$\varepsilon = \frac{\delta m}{m_1} = \int_0^\infty \gamma(t)\,\mathrm{d}t \tag{2.11}$$

whence one readily verifies the following relations:

$$\delta m = \tfrac{4}{3} m_{\mathrm{e}} \tag{2.12a}$$

$$m = m_1(1+\varepsilon) \tag{2.12b}$$

$$\int_0^\infty t\gamma(t)\,\mathrm{d}t = 2e^2/3m_1c^3 = (1+\varepsilon)\tau_0 \tag{2.12c}$$

τ_0 having its usual meaning.

Some preliminary features of the model deserve attention. It is clear that, for a given family of charge distributions containing a radius parameter, the electrostatic energy m_{e} grows with decreasing value of this parameter. In other words, denoting the radius of the electron by r_{e}, we can say that, if $r_{\mathrm{e}} \to \infty$, then $m_{\mathrm{e}} \to 0$, and if $r_{\mathrm{e}} \to 0$, $m_{\mathrm{e}} \to \infty$.

Consequently, according to (2.7), (2.10) and (2.12*a*) and taking into account that m has a well defined value, we easily conclude that m_1 takes values between $-\infty$ for $r_{\mathrm{e}} \to 0$ ($m_1 \sim -\delta m$) and m for $r_{\mathrm{e}} \to \infty$. We can select two characteristic radii, namely a first one that we call the critical radius, r_{cr}, for which

$$m = \tfrac{4}{3} m_{\mathrm{e}}(r_{\mathrm{cr}}) \Leftrightarrow m_1(r_{\mathrm{cr}}) = 0 \tag{2.13}$$

and a second one, r_{u}, satisfying

$$\varepsilon(r_{\mathrm{u}}) = 1 \Leftrightarrow \tfrac{8}{3} m_{\mathrm{e}}(r_{\mathrm{u}}) = m. \tag{2.14}$$

While the interest of the latter will appear clear in the following sections, some peculiarities of the former are already obvious. According to the above comments, it is clear from the definition of r_{cr} and equations (2.7), (2.10) and (2.12*a*) that m_1 is positive for $r_{\mathrm{e}} > r_{\mathrm{cr}}$ and negative for $r_{\mathrm{e}} < r_{\mathrm{cr}}$.

Now, if we consider that the external force sets in at t_0 and the electron has a constant velocity for $t < t_0$, the equation of motion (2.9) is written

$$\ddot{\boldsymbol{r}} = 0 \qquad t < t_0 \tag{2.15a}$$

$$\ddot{\boldsymbol{r}} = \frac{\boldsymbol{F}}{m_1} - \int_{t_0}^{t} \mathrm{d}t'\,\gamma(t-t')\ddot{\boldsymbol{r}}(t') \qquad t \geqslant t_0. \tag{2.15b}$$

For times larger than but very close to t_0, we see that the motion is ruled by the first term on the RHS of equation (2.15*b*), i.e.

$$\ddot{\boldsymbol{r}} \simeq \boldsymbol{F}/m_1. \tag{2.16}$$

This means that m_1 represents the initial inertia to the motion. Two conclusions arise from this. First, if the radius has a value very close to the critical one, m_1 is very small and the charge can attain a high speed quickly, because of which the non-relativistic approximation can be erroneous. Second, if $r_e < r_{cr}$, the initial inertia m_1 is negative. We must point out that this behaviour can be considered exact at the first instants, and not a consequence of the approximations, for a rigid spherically symmetric distribution (see França *et al* (1978) for details concerning the derivation of equation (2.3)). This is a very surprising result that escapes any physical intuition. Obviously it demands a deeper specific analysis which will be the subject of a future work.

The value of the critical radius depends on the model. However, an estimation can be made. If we admit that the distribution of the charge is rather uniform within a volume of radius r_e, we can estimate

$$m_e \sim \frac{e^2}{2c^2}\frac{1}{r_e}. \tag{2.17}$$

Consequently, from (2.12*a*) and (2.13) we obtain

$$\tfrac{3}{4}m = m_e(r_{cr}) \sim \frac{e^2}{2c^2}\frac{1}{r_{cr}} \Rightarrow r_{cr} \sim c\tau_0 = 1.88\ F. \tag{2.18}$$

In other words, the critical radius is, in general, of the order of the so-called classical radius.

I also want to draw attention to the r_e dependence of the parameter ε. It is easy to see from the behaviour of m_1 and (2.12*b*) that ε is positive for $r_e > r_{cr}$ with values between 0 for $r_e \to \infty$ and $+\infty$ for $r_e \to r_{cr^+}$, and, for $r_e < r_{cr}$, $\varepsilon < -1$ going to $-\infty$ for $r_e \to r_{cr}$.

To end this section, a point concerning the charge model should be pointed out. A disagreement in the equation of motion can be found in some papers (see, e.g., Bohm and Weinstein 1948), in that an amount $-\frac{1}{3}m_e$ is missing in the computation of the mass. As a consequence, the equation of motion is written like (2.9) but with m_0 instead of m_1 in equations (2.8) and (2.9). In this situation $m = m_0 + \frac{4}{3}m_e$ and the features appearing in our model for $m_1 < 0$ correspond in these other models to $m_0 < 0$. That is, the critical radius for these models is the minimum for which m_0 is non-negative. As m_0 is the mechanical mass, if we admit that it cannot be negative we arrive at the conclusion that in these models the radius is never smaller than the critical one. On the contrary, in our model, $m_0 = m_1 + \frac{1}{3}m_e$ whence it is possible to have $m_1 < 0$ with $m_0 > 0$, and then radii smaller than the critical value are allowed. This will be important in the results of the present paper.

3. Main result: uniqueness of the solution starting from a finite time

In the following we consider that the force is switched on at a time $t = 0$. The problem of the unicity of solutions requires to be analysed for both $t < 0$ and $t > 0$. As we shall see, the multiplicity of solutions is due, in general, to the former. To begin this analysis we devote this section to prove that, in very general conditions, the solution for $t > 0$ is unique. Specifically, we shall prove the following.

Theorem. If the following conditions are satisfied:

(i) $\boldsymbol{r}(t)$ is known for $t<0$,

(ii) $\dot{\gamma}(t)$ exists and is bounded at least in an interval to the right of the origin, $[0, t_a)$, and

(iii) the external force $\boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v}, t)/m_1$ is a continuous function of its arguments with first space and velocity derivatives, except, perhaps, in isolated points, and such that in every bounded phase-space interval such derivatives are all bounded,

then the solution $\boldsymbol{r}(t)$ of equation (2.9) for $t>0$ is unique.

Note that there is no restriction upon the values of the radius, with the exception of r_{cr}.

The first condition reduces the problem of unicity to the solutions of the homogeneous equation. The second one is shown in the appendix to be valid for very general and physically reasonable charge distributions. Finally, condition (iii) may be considered necessary for every physical force. Consequently, we deduce that the three conditions are completely general from a physical point of view.

Proof. We begin by writing the equation of motion in the form

$$\ddot{\boldsymbol{r}}(t)=\boldsymbol{f}(\boldsymbol{r}, t)+\boldsymbol{\varphi}(t)-\int_0^t \gamma(t-t')\ddot{\boldsymbol{r}}(t')\,\mathrm{d}t' \tag{3.1}$$

where

$$\boldsymbol{f}(\boldsymbol{r}, t)=\boldsymbol{F}(\boldsymbol{r}, t)/m_1 \tag{3.2a}$$

$$\boldsymbol{\varphi}(t)=-\int_{-\infty}^0 \gamma(t-t')\ddot{\boldsymbol{r}}(t')\,\mathrm{d}t'. \tag{3.2b}$$

Since only the motion at negative values of the time contributes in $\boldsymbol{\varphi}$, this is a known function of the time.

Let us now suppose that there exists a time t_0 and two solutions of (3.1), $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$, such that

$$\boldsymbol{u}(t)\equiv\boldsymbol{r}_2(t)-\boldsymbol{r}_1(t)\equiv 0 \qquad \text{for } t\leqslant t_0 \tag{3.3a}$$

$$\boldsymbol{u}(t)\equiv\boldsymbol{r}_2(t)-\boldsymbol{r}_1(t)\neq 0 \qquad \text{for } t>t_0. \tag{3.3b}$$

We may assume without loss of generality that in a first time interval $[t_0, t_1)$, both $|\dot{\boldsymbol{u}}|$ and $|\ddot{\boldsymbol{u}}|$ are strictly increasing functions.

Now let

$$\boldsymbol{I}_i(t)=\ddot{\boldsymbol{r}}_i(t)-\boldsymbol{f}(\boldsymbol{r}_i(t), t)-\boldsymbol{\varphi}(t)+\int_{t_0}^t \gamma(t-t')\ddot{\boldsymbol{r}}_i(t')\,\mathrm{d}t' \qquad i=1,2 \tag{3.4}$$

and

$$\begin{aligned}0\equiv B&=\int_{t_0}^{t_1}\mathrm{d}t(t_1-t)[\boldsymbol{I}_2(t)-\boldsymbol{I}_1(t)]\cdot\dot{\boldsymbol{u}}(t)\\ &\equiv\int_{t_0}^{t_1}\mathrm{d}t(t_1-t)\Big(\ddot{\boldsymbol{u}}(t)\cdot\dot{\boldsymbol{u}}(t)-\{\boldsymbol{f}_2-\boldsymbol{f}_1\}\cdot\dot{\boldsymbol{u}}(t)\\ &\quad+\int_{t_0}^{t}\mathrm{d}t'\,\gamma(t-t')\ddot{\boldsymbol{u}}(t')\cdot\dot{\boldsymbol{u}}(t)\,\mathrm{d}t'\Big)\end{aligned} \tag{3.5}$$

with

$$f_i = f(r_i(t), t) \qquad i = 1, 2.$$

Elementary algebra yields

$$0 \equiv B = \tfrac{1}{2}\int_{t_0}^{t_1} \mathrm{d}t\, \dot{u}^2(t) + \int_{t_0}^{t_1} \mathrm{d}t \int_{t_0}^{t} \mathrm{d}t'\, \dot{\gamma}(t-t')\dot{u}(t)\cdot\dot{u}(t')(t_1 - t)$$
$$+ \int_{t_0}^{t_1} \mathrm{d}t\,(t_1 - t)\dot{u}(t)\cdot\{f_1 - f_2\}. \tag{3.6}$$

Now, taking into account condition (ii) and the increasing behaviour of $|\dot{u}(t)|$, and choosing t_1 such that $|t_1 - t_0| \leqslant t_a$, we can write

$$\left|\int_{t_0}^{t_1} \mathrm{d}t \int_{t_0}^{t} \mathrm{d}t'\, \dot{\gamma}(t-t')\dot{u}(t')\cdot\dot{u}(t)(t_1 - t)\right|$$
$$\leqslant \sup_{t\in[0,t_1-t_0]} |\dot{\gamma}(t)| \int_{t_0}^{t_1} \mathrm{d}t \int_{t_0}^{t} \mathrm{d}t' |\dot{u}(t)|^2 |t_1 - t_0|$$
$$\leqslant \sup_{t\in[0,t_1-t_0]} |\dot{\gamma}(t)| \int_{t_0}^{t_1} \mathrm{d}t\, \dot{u}^2(t)(t_1 - t_0)^2 \tag{3.7}$$

and then

$$\int_{t_0}^{t_1} \mathrm{d}t \int_{t_0}^{t} \mathrm{d}t'\, \dot{\gamma}(t-t')\dot{u}(t)\cdot\dot{u}(t')(t_1 - t) \geqslant - \sup_{t\in[0,t_1-t_0]} |\dot{\gamma}(t)| \int_{t_0}^{t_1} \dot{u}^2(t)\,\mathrm{d}t\,(t_1 - t_0)^2. \tag{3.8}$$

As concerns the last term of (3.6), if we call V_0 a bounded phase-space interval containing both $(r(t_0), v(t_0))$ and the paths $(r_1(t), v_1(t))$ and $(r_2(t), v_2(t))$ between t_0 and t_1, we can write

$$\int_{t_0}^{t_1} \mathrm{d}t\,(t_1 - t)\dot{u}(t)\cdot\{f_1 - f_2\}$$
$$= \int_{t_0}^{t_1} \mathrm{d}t\,(t_1 - t)\dot{u}(t)\cdot\left(\sum_j \frac{\partial f}{\partial x_j}\bigg|_{(r_I, v_I, t)} u_j(t) + \sum_j \frac{\partial f}{\partial v_j}\bigg|_{(r_I, v_I, t)} \dot{u}_j(t)\right) \tag{3.9}$$

(r_I, v_I) being a particular point in V_0. If M and M' denote bounds for, respectively, all $\partial f_i/\partial x_j$ and $\partial f_i/\partial v_j$ in V_0, we obtain

$$\left|\int_{t_0}^{t_1} \mathrm{d}t\,(t_1 - t)\dot{u}(t)\cdot\{f_1 - f_2\}\right| \leqslant \int_{t_0}^{t_1} \mathrm{d}t(t_1 - t)\sum_{ij} |\dot{u}_i(t)|$$
$$\times [M|u_j(t)| + M'|\dot{u}_j(t)|]$$
$$\leqslant 9\int_{t_0}^{t_1} \mathrm{d}t\,(t_1 - t)[M|\dot{u}(t)|\,|u(t)| + M'|\dot{u}(t)|^2]. \tag{3.10}$$

Now

$$|u(t)| = \left|\int_{t_0}^{t} \mathrm{d}t'\, \dot{u}(t')\right| \leqslant \int_{t_0}^{t} \mathrm{d}t'\, |\dot{u}(t')| \leqslant |\dot{u}(t)|(t - t_0) \tag{3.11}$$

because $|\dot{u}|$ is an increasing function. Then

$$\left|\int_{t_0}^{t_1} \mathrm{d}t(t_1 - t)\dot{u}(t)\cdot\{f_1 - f_2\}\right| \leqslant 9[M(t_1 - t_0)^2 + M'(t_1 - t_0)] \int_{t_0}^{t_1} \mathrm{d}t\, |\dot{u}(t)|^2 \tag{3.12}$$

and consequently

$$\int_{t_0}^{t_1} \mathrm{d}t\,(t_1-t)\dot{u}(t)\cdot\{f_1-f_2\}\geqslant -9\int_{t_0}^{t_1}\mathrm{d}t\,|\dot{u}(t)|^2[M(t_1-t_0)^2+M'(t_1-t_0)]. \tag{3.13}$$

Now, introducing (3.8) and (3.13) into (3.6) we obtain

$$0\equiv B\geqslant \tfrac{1}{2}\int_{t_0}^{t_1}\mathrm{d}t\,\dot{u}^2(t)$$
$$\times\left[1-2\sup_{t\in[0,t_1-t_0]}|\dot{\gamma}(t)|(t_1-t_0)^2-18\{M(t_1-t_0)^2+M'(t_1-t_0)\}\right]. \tag{3.14}$$

Obviously we may choose t_1 such that (t_1-t_0) is sufficiently small so that the RHS of (3.14) is strictly positive. Consequently it is necessary that $\dot{u}\equiv 0$ in $[t_0, t_1)$, in contradiction with our assumption that $\dot{u}\neq 0$ starting from t_0. This ends the proof of the theorem.

According to this theorem, if we know $r(t)$ for $t<0$, the subsequent trajectory is determined by (2.9). However, this equation, written now as (3.1), clearly depends on $r(t)$ for $t<0$ through the term $\varphi(t)$. It is obvious that $r\equiv 0$ is a solutuion of (2.9) if $F\equiv 0$, and then $\varphi(t)$ would be zero in (3.1). However, if other solutions of the homogeneous version of (3.1) exist, other functions $\varphi(t)$ will have to be considered.

This means that the uniqueness of solutions of (2.9) solely depends of the uniqueness of solutions $a(t)$ for its homogeneous version

$$a(t)=-\int_{-\infty}^{t}\mathrm{d}t'\,\gamma(t-t')a(t') \tag{3.15}$$

$a(t)$ being the acceleration.

A final remark is necessary. If we admit that the force is turned on at $t=-\infty$, the foregoing result does not solve the problem of unicity. However, physical intuition makes us hope that in this case the solutions are related to the asymptotic behaviour of those obtained when the force sets in at a finite time and, consequently, the solution of the unicity problem would again be related to the analysis of the homogenous equation, equation (3.15).

Nevertheless this case requires a specific study that we do not develop in this article.

4. Solutions of the homogenous equation

According to the preceding section, the uniqueness problem for equation (2.3) is in fact the one corresponding to its homogeneous version, equation (3.15). Although not complete, a general answer to this has already been given.

On the one hand, it has recently (Blanco *et al* 1986) been proven that if the sign of $\gamma(t)$ is constant, for $|\varepsilon|<1$, i.e. for $r_e>r_u$, no solutions $a(t)$ that are bounded at $t\to-\infty$ exist, except the trivial one. (This result slightly generalises an older one (Alvarez-Estrada and Ros Martínez 1981) which only considers solutions with vanishing acceleration at $t\to-\infty$.)

On the other hand, some non-trivial solutions of (3.15) have often been found in the literature for $|\varepsilon|>1$, i.e. for $r_e<r_u$ (see for instance, Blanco *et al* 1986, França *et al* 1978, Moniz and Sharp 1977). Obviously the existence of these solutions is not

general and depends on the charge distribution. The solutions mentioned are of exponential type, $e^{\mu t}$. For them we can say the following.

(*a*) Only solutions with either real ($\mu \in R$) or purely imaginary ($i\mu \in R$) exponents exist.

(*b*) In the first case there are no solutions if $\varepsilon > 0$, i.e. if $r_e > r_{cr}$.

(*c*) The other case gives rise to so-called self-oscillations and is possible for $r_e < r_u$, if the two following conditions are satisfied ($i\mu \in R$):

$$\hat{\rho}(i\mu) = 0 \tag{4.1a}$$

$$1 + 16\pi^2\tau_0(1+\varepsilon)\,\mathrm{VP}\int_0^\infty d\omega \frac{\omega^2}{\omega^2+\mu^2}\hat{\rho}^2(\omega) = 0 \tag{4.1b}$$

(where VP denotes the principal value).

Nevertheless, it is clear that all these results concerning the homogeneous equation only give a partial answer to the problem of the general solution of (3.15). For instance, it is not known whether other solutions that are not linear combinations of the exponentials above considered are possible. This point will not be considered further in the present paper and will be the subject of future work. Here we seek to call attention to the fact that the most general solution of (3.15) has not yet been given.

5. Pre-acceleration and runaways

In this section we seek to clarify a few points concerning these two phenomena.

5.1. Runaways

The behaviour of the charge obviously depends on the sort of force acting upon it. Thus, if such a force continuously gives energy to the particle, it will certainly display a runaway behaviour. This kind of motion is pathological only when the force cannot give such an amount of energy to the charge. This is why it is usual to restrict the search for such an undesirable runaway behaviour to the two following general situations:

(i) the motion after the external force is turned off, and

(ii) the motion under a conservative force field.

We find an analysis of the former only in Kaup (1966). However, the proof displayed in this work specifically uses the motion for $t < 0$ as produced by a position-independent force. Consequently, the problem of runaways is not solved in general. The reason why we consider this phenomenon appears in connection with the other problem of this section.

5.2. Pre-acceleration

To analyse this problem we assume that the force is turned on at $t = 0$. The equation of motion now becomes

$$\ddot{r}(t) = -\int_{-\infty}^{t} \gamma(t-t')\dot{r}(t')\,dt' \qquad t<0 \tag{5.1a}$$

$$\ddot{r}(t) = \boldsymbol{F} + \boldsymbol{\phi}(t) - \int_0^t \gamma(t-t')\dot{r}(t')\,dt' \qquad t>0 \tag{5.1b}$$

where $\boldsymbol{\phi}(t)$, given by

$$\boldsymbol{\phi}(t) = -\int_{-\infty}^{0} \gamma(t-t')\ddot{\boldsymbol{r}}(t')\,\mathrm{d}t' \tag{5.2}$$

carries the effect of memory due to the motion before $t=0$.

It is trivial from (5.1a, b) to see that the solution for $t<0$, if it exists, is independent of the force. Moreover, if $\boldsymbol{F}$ is only time dependent, and admits a Laplace transform, we may write the solution, at $t>0$, as

$$\ddot{\boldsymbol{r}}(t) = \int_0^{\infty} \mathrm{d}t'\,[\boldsymbol{\phi}(t') + \boldsymbol{F}(t')]\chi(t-t') \tag{5.3}$$

with $\chi(t)$ given by

$$\chi(t) = \frac{1}{2\pi\mathrm{i}} \int_{C\uparrow} \frac{\mathrm{e}^{zt}}{1+\tilde{\gamma}(z)}\,\mathrm{d}z \tag{5.4}$$

where $\tilde{f}(z)$ stands for the Laplace transform of f,

$$\tilde{f}(z) = \int_0^{\infty} \mathrm{d}t\,\mathrm{e}^{-zt} f(t) \tag{5.5}$$

and C denotes a vertical line in the complex plane which lies to the right of σ_c, the abscissa of convergence of $(\tilde{\boldsymbol{F}}+\tilde{\boldsymbol{\phi}})/(1+\tilde{\gamma})$. It is easy to see that σ_c is finite. Firstly, we have assumed that $\boldsymbol{F}$ admits a Laplace transform. Secondly, $\tilde{\boldsymbol{\phi}}(z)$ exists even for $\operatorname{Re} z = 0$ if we assume $\boldsymbol{a}(t)$ to be bounded at $t\to-\infty$ (which is physically reasonable) because then $\boldsymbol{\phi}(t)$ decreases with time in a similar way to $\gamma(t)$. Finally, for $\operatorname{Re} z$ big enough zeros of $1+\tilde{\gamma}$ cannot exist. This is due to the fact that $\tilde{\gamma}$ tends to zero as $\operatorname{Re} z \to \infty$, and

$$|\tilde{\gamma}(z)| \leq \tilde{\gamma}(\operatorname{Re} z). \tag{5.6}$$

Consequently, σ_c is finite. Now, if we close the contour of integration to the right of the complex plane for $t<0$, it is easy to obtain

$$t<0 \Rightarrow \chi(t) = 0 \tag{5.7}$$

and then equation (5.3) becomes

$$\ddot{\boldsymbol{r}}(t) = \int_0^{t} \mathrm{d}t'\,[\boldsymbol{\phi}(t') + \boldsymbol{F}(t')]\chi(t-t') \tag{5.8}$$

which shows that $\ddot{\boldsymbol{r}}(t)$ is due only to the force at times prior to t.

Consequently we can say that our model does not show pre-acceleration.

This result seems to be in contradiction to statements made by França *et al* (1978) and de la Peña *et al* (1982) where the phenomenon of pre-acceleration is claimed to exist for radii sufficiently small. We shall show that our disagreement in fact is a matter of interpretation.

Let us consider a time-dependent force $\boldsymbol{F}(t)$ that sets in at $t=0$. If we look for the solution using the Fourier transform we obtain

$$\ddot{\boldsymbol{r}}(t) = \ddot{\boldsymbol{r}}_0(t) + \int_0^{\infty} \mathrm{d}t'\,\boldsymbol{F}(t')G(t-t') \tag{5.9}$$

where $\ddot{r}_0$ is a solution of the homogeneous equation,

$$G(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\mu \frac{\mathrm{e}^{-\mathrm{i}\mu t}}{1+\hat{\gamma}_0(\mu)} \tag{5.10}$$

and

$$\hat{\gamma}_0(\mu)=\int_{-\infty}^{\infty} \gamma_0(t)\,\mathrm{e}^{\mathrm{i}\mu t}\,\mathrm{d}t=\int_0^{\infty} \gamma(t)\,\mathrm{e}^{\mathrm{i}\mu t}\,\mathrm{d}t=\tilde{\gamma}(-\mathrm{i}\mu). \tag{5.11}$$

This is the method followed in the aforementioned papers. However, it must be pointed out that (5.9) is valid only if $1+\hat{\gamma}_0(\mu)$ does not have zeros on the real axis. Otherwise, a few changes have to be introduced, but the analysis is similar to the following. Let us then assume that (5.10) is valid. The authors of the quoted papers take $\ddot{r}_0\equiv 0$. The following step is now the analysis of $G(t)$: pre-acceleration (acausality) exists if and only if $G(t)\neq 0$ for $t<0$—as is easily seen in (5.9)—i.e. if $1+\hat{\gamma}_0(\mu)$ has zeros on the upper half-plane. Using (5.11) this means that $1+\tilde{\gamma}(p)$ has zeros for $\operatorname{Re} p>0$. However, an analysis similar to that used by de la Peña *et al* (1982) shows that this is only possible if $m_1<0$ and $\operatorname{Im} p=0$, i.e. the zeros of $(1+\tilde{\gamma}(p))$ with positive real part, if any, lie on the real axis. Consequently, we conclude that for $m_1<0$, i.e. $r_e<r_{cr}$, pre-acceleration exists.

However, (5.9) with the condition $\ddot{r}_0=0$ is not a general solution of the motion, but a particular one. The point is that, as we shall show in the following, the part of $G(t)$ that presumably gives rise to the pre-acceleration in fact gives rise to a solution of the homogeneous equation. To see this note that, as $\tilde{\gamma}(p)$ is an analytic function for $\operatorname{Re} p>0$, the zeros of $1+\hat{\gamma}_0(\mu)$ are isolated points.

Furthermore, for the reason mentioned in relation to (5.6), all zeros lie in a bounded interval and then there are a finite number of them. Finally, these zeros are of order 1, i.e. $\hat{\gamma}_0'(\mu)\neq 0$. To see this, note that

$$\hat{\gamma}_0'(\mu)=-\mathrm{i}\tilde{\gamma}'(-\mathrm{i}\mu) \tag{5.12}$$

and for the zeros, $\mu_j=\mathrm{i}\lambda_j$ with $\lambda_j>0$, we have

$$\hat{\gamma}_0'(\mathrm{i}\lambda_j)=-\mathrm{i}\tilde{\gamma}'(\lambda_j)=\mathrm{i}\int_0^{\infty} \mathrm{d}t\,\gamma(t)\,\mathrm{e}^{-\lambda_j t} \tag{5.13}$$

and the last integral is strictly positive.

Going back to (5.10), all statements made so far amount to saying that there are a finite number of simple poles with $\operatorname{Im}\mu>0$ all on the imaginary axis. Now we consider a rectangle C in the complex plane with vertices $-M$, M, $M+\mathrm{i}P$, $-M+\mathrm{i}P$, P being such that all zeros of $1+\hat{\gamma}_0(\mu)$ lie inside the rectangle. We now integrate the integrand in (5.10) over C. The integral over the vertical lines goes to zero when $M\to+\infty$. To see this, note that $\hat{\gamma}_0(\mu_1+\mathrm{i}\mu_2)$ goes to zero if $\mu_1\to\infty$. Then $\forall\varepsilon>0\ \exists\mu_0/\ \forall\mu_1>\mu_0$, $|\hat{\gamma}_0(\mu_1+\mathrm{i}\mu_2)|<\varepsilon$ and therefore

$$\begin{aligned}&\left|\int_0^P \mathrm{d}\mu_2 \frac{\exp[-\mathrm{i}(\mu_1+\mathrm{i}\mu_2)t}{1+\hat{\gamma}_0}-\int_0^P \mathrm{d}\mu_2 \exp[-\mathrm{i}(\mu_1+i\mu_2)t]\right|\\ &\quad=\left|\int_0^P \mathrm{d}\mu_2 \frac{\hat{\gamma}_0}{1+\hat{\gamma}_0}\exp[-\mathrm{i}(\mu_1+\mathrm{i}\mu_2)t]\right|\leqslant\frac{\varepsilon}{|1-\varepsilon|}\int_0^P \mathrm{d}\mu_1\to 0\end{aligned} \tag{5.14}$$

$$\int_0^P \mathrm{d}\mu_2 \exp[-\mathrm{i}(\mu_1+\mathrm{i}\mu_2)t]=\frac{1-\exp[-\mathrm{i}(\mu_1+\mathrm{i}p)t]}{\mathrm{i}t(\mu_1+\mathrm{i}\mu_2)}\xrightarrow[\mu_1\to\infty]{} 0. \tag{5.15}$$

Consequently we may write, calling C_1 the horizontal upper side of the rectangle,

$$G(t) = G_0(t) + \mathrm{i} \sum_j K_j \, \mathrm{e}^{\lambda_j t} \tag{5.16}$$

where

$$G_0(t) = \frac{1}{2\pi} \int_{C_1\rightarrow} \mathrm{d}\mu \frac{\mathrm{e}^{-\mathrm{i}\mu t}}{1 + \hat{\gamma}_0(\mu)} \tag{5.17}$$

and the K_j are the residues of $(1 + \hat{\gamma}_0(\mu))^{-1}$ in each pole λ_j,

$$K_j = \frac{1}{\hat{\gamma}_0'(\mu)|_{\mu = \mathrm{i}\lambda_j}} = \frac{1}{-\mathrm{i}\tilde{\gamma}'(\lambda_j)}. \tag{5.18}$$

The function $G_0(t)$ is now a causal function, i.e. $t < 0 \Rightarrow G_0(t) = 0$. This is clear because, by construction, there are no poles in the half-plane over C_1.

From (5.16)-(5.18), equation (5.9) can be written

$$\ddot{\boldsymbol{r}}(t) = \ddot{\boldsymbol{r}}_0(t) + \sum_j \boldsymbol{M}_j \, \mathrm{e}^{\lambda_j t} \qquad t < 0 \tag{5.19a}$$

$$\ddot{\boldsymbol{r}}(t) = \ddot{\boldsymbol{r}}_0(t) + \sum_j \boldsymbol{M}_j \, \mathrm{e}^{\lambda_j t} - \int_0^t \mathrm{d}t' \, \boldsymbol{F}(t') G_0(t - t') \qquad t > 0 \tag{5.19b}$$

where

$$\boldsymbol{M}_j = -\int_0^\infty \mathrm{d}t' \frac{\boldsymbol{F}(t') \, \mathrm{e}^{-\lambda_j t'}}{\tilde{\gamma}'(\lambda_j)}. \tag{5.20}$$

The crucial point is that all the exponentials $\mathrm{e}^{\lambda_j t}$ are solutions of the homogeneous equation, which can be seen by substitution and taking into account that

$$1 + \tilde{\gamma}(\lambda_j) = 0. \tag{5.21}$$

We see now that choosing $\ddot{\boldsymbol{r}}_0(t) = 0$ defines a particular, but not a general, solution for the motion. However, there is nothing in the model that prescribes such a choice. In consequence, the model does not show pre-acceleration but, for $m_1 < 0$, an 'anomalous' indetermination of solutions for the free case instead. (Recall that this indetermination arises in the homogeneous equation.)

There is an interesting relation between the behaviour before $t = 0$ and the behaviour for $t \to \infty$ which reminds us of the LD equation when all its solutions are considered. To see it, let us suppose that $\boldsymbol{F}(t)$ is turned off at $t = t_1$. We have, from (5.9) and (5.19a),

$$\ddot{\boldsymbol{r}}(t) = \ddot{\boldsymbol{r}}_0(t) + \sum_j \boldsymbol{M}_j \, \mathrm{e}^{\lambda_j t} \qquad t < 0 \tag{5.22a}$$

$$\ddot{\boldsymbol{r}}(t) = \ddot{\boldsymbol{r}}_0(t) - \int_0^{t_1} \mathrm{d}t' \, F(t') G(t - t') \qquad t > 0. \tag{5.22b}$$

For $t > t_1 > t'$ in the last integral G can be calculated from (5.10) closing the contour of integration through the lower half-plane. Consequently $G(t)$ is bounded and that term does not show runaway behaviour. However, the total solution depends on the determination of $\ddot{\boldsymbol{r}}_0(t)$. What we want to exhibit here is that if one prescribes $\ddot{\boldsymbol{r}}_0 = 0$, then the solution does not show runaway behaviour but it shows pre-acceleration. However, we can eliminate the latter, choosing

$$\ddot{\boldsymbol{r}}_0(t) = -\sum_j \boldsymbol{M}_j \, \mathrm{e}^{\lambda_j t} \tag{5.23}$$

but then $\ddot{r}$ displays runaway behaviour. Finally, both phenomena could appear if a different choice of $\ddot{r}_0(t)$ is made.

We conclude this section by noting that this indetermination, which appears when the radius of the charge is smaller than the critical radius, seems to us to be pathological and undesirable and consequently a deeper analysis of this problem seems necessary.

6. Conclusions and discussion

We have shown that the model of extended charge we have analysed is essentially causal: on the one hand the motion before a certain time t_0 and the system of forces uniquely determine the motion after t_0; on the other hand the force at a time t_1 influences the motion only at posterior times $t > t_1$. Moreover, this is true for any value of the radius of the charge.

However, some anomalous behaviour is detected if the radius is not larger than a certain value r_u. While for $r > r_u$, the solution is unique for each value of the position and velocity at the time when the force is turned on, for $r \leq r_u$ there is indetermination in the solution due to an indetermination in the absence of a force, i.e. for the homogeneous equation. Nevertheless, the general solutions to this have not been studied up to now. Only some particular solutions, of the form $e^{\mu t}$, are known. The question arises whether these are the only possible solutions.

Another open problem is that of runaway behaviour. This is partially related to the solutions of the homogeneous equation. In this case also, only partial results are known.

The non-generality of all these results prevents us from giving a complete interpretation of the model. We intend to devote future work to these subjects.

Finally, we must recall the pathological behaviour at the first instants of the action of the force if $r < r_{cr}(< r_u)$, where the initial inertia turns out to be negative. This problem deserves special attention, which it has not received up to now.

Appendix

We want to show that for very general conditions, $\dot{\gamma}(t)$ exists and is bounded. We shall assume the following conditions:

(i) There exists the limit

$$L = \lim_{r \to 0+} r\rho(r). \tag{A1}$$

(ii) $d(r\rho(r))/dr$ is absolutely integrable in every finite or infinite interval.

The first condition is in fact weaker than what is expected from a real charge distribution. In a realistic case ρ should be finite at $r = 0$ and, moreover, $\rho'(0) = 0$.

The second condition is

$$\int_a^b |\rho(r) + r\rho'(r)| \, dr < \infty \qquad \forall a, b > 0 \tag{A2}$$

where a and b could be ∞. This condition is fulfilled, for instance, by a charge density which is piecewise differentiable and monotonically decreasing for r sufficiently large.

To see this, let R be such that for $r > R$ $\rho' < 0$. Then with $b > R$

$$\int_a^b |\rho(r) + r\rho'(r)|\,\mathrm{d}r < \int_a^b \rho(r)\,\mathrm{d}r + \int_a^R |r\rho'(r)|\,\mathrm{d}r - \int_R^b r\rho'(r)\,\mathrm{d}r. \tag{A3}$$

It is immediate to see that all these integrals are finite even for $b = \infty$.

However, it is not clear to what extent condition (ii) is necessary, but we have the impression that it is not. Obviously, in this case the analysis should be different.

Let us now prove that $\dot{\gamma}$ exists and is bounded. For this we study first the asymptotic behaviour of $\hat{\rho}(\omega)$. From its definition, expression (2.1), and taking into account the properties of ρ, elementary algebra yields

$$\hat{\rho}(\omega) = (2/\pi)^{1/2}(1/k)f(k)|_{k=\omega/c} \tag{A4}$$

with

$$f(k) = \int_0^\infty \mathrm{d}r\, r\rho(r)\sin kr. \tag{A5}$$

Let us consider the asymptotic behaviour of $f(k)$. We write it as

$$\begin{aligned} f(k) &= -1/k \int_0^\infty r\rho(r)\,\mathrm{d}(\cos kr) = -\frac{1}{k}\left((r\rho\cos kr)|_0^\infty - \int_0^\infty \cos kr(r\rho' + \rho)\,\mathrm{d}r\right) \\ &= \frac{L}{k} + \frac{1}{k}\int_0^\infty \cos kr(r\rho' + \rho)\,\mathrm{d}r. \end{aligned} \tag{A6}$$

Now

$$|f(k)| \leq \frac{L + \int_0^\infty |r\rho' + \rho|\,\mathrm{d}r}{k}. \tag{A7}$$

Consequently, when $k \to \infty$, $f(k)$ behaves at most as $1/k$.

$$\lim_{k\to\infty} kf(k) < \infty \tag{A8}$$

and from (A4) $\hat{\rho}(\omega)$ behaves at most as $1/\omega^2$,

$$\lim_{\omega\to\infty} \omega^2\hat{\rho}(\omega) < \infty. \tag{A9}$$

Going now to the function $\gamma(t)$

$$\gamma(t) = \text{constant} \times \int_0^\infty \mathrm{d}\omega\, \omega\hat{\rho}^2 \sin \omega t \tag{A10}$$

from (A9) it results that

$$\int_0^\infty \omega^2\hat{\rho}^2(\omega)\,\mathrm{d}\omega < \infty \tag{A11}$$

and consequently the theorems of the analysis enable us to write

$$\dot{\gamma}(t) = \text{constant} \times \int_0^\infty \omega^2\hat{\rho}^2(\omega)\cos\omega t\,\mathrm{d}\omega. \tag{A12}$$

Moreover

$$|\dot{\gamma}(t)| \leq |\text{constant}| \int_0^\infty \omega^2\hat{\rho}^2(\omega)\,\mathrm{d}\omega = |\dot{\gamma}(0)| \tag{A13}$$

for all t.

References

Alvarez-Estrada R F and Ros Martínez E 1981 *An. Fís.* A **77** 110
Blanco R, Pesquera L and Jiménez J L 1986 *Phys. Rev.* D **34** 452
Bohm D and Weinstein M 1948 *Phys. Rev.* **74** 1789
Caldirola P 1956 *Suppl. Nuovo Cimento* **3** 297
Daboul J 1975 *Z. Phys.* B **21** 115
Daboul J and Jensen J H D 1973 *Z. Phys.* **265** 455
de la Peña L, Jiménez J L and Montemayor R 1982 *Nuovo Cimento* B **69** 71
Erber T 1961 *Fortschr. Phys.* **9** 343
França H M, Marques G C and da Silva A J 1978 *Nuovo Cimento* A **48** 65
Grandy W T and Aghazadeh A 1982 *Ann. Phys., NY* **142** 284
Hirosige T 1965 *Japan Stud. Hist. Sci.* **5** 1
Jiménez J L, Blanco R and Pesquera L 1985 *J Phys. A: Math. Gen* **18** 1173
Kaup D J 1966 *Phys. Rev.* **152** 1130
Levine H, Moniz E J and Sharp D H 1977 *Am. J. Phys.* **45** 75
Lorentz H A 1952 *The Theory of Electrons* (New York: Dover)
Markov M 1946 *J. Phys. (USSR)* **10** 159
Miller A I 1976 *Am. J. Phys.* **44** 912
Moniz E J and Sharp D H 1977 *Phys. Rev.* D **15** 2850
Nodvik J S 1964 *Ann. Phys., NY* **28** 225
Radaña A F and Vázquez L 1984 *J. Phys. A: Math. Gen.* **17** 2011